Non-Linear Predictive Control of Multi-Input Multi-Output Vehicle Suspension System

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ABSTRACT

A new predictive control approach is employed to control a non-linear 4 degrees of freedom vehicle suspension model including pitch and bounce motions. The front and rear suspension actuator forces are considered as control inputs. These are determined by minimizing a performance index defined as a weighted combination of conflicting objectives, namely ride comfort and road holding. In this way, the suspension deflection and the control forces should be restricted to admissible practical ranges. In this paper, an optimization process is used to develop a multi-input multi-output (MIMO) non-linear control law in a closed-form which is mathematically tractable and suitable for implementation. To investigate special features of the derived control law, the closed-loop system dynamics are first evaluated analytically and then the corresponding simulation results are presented. The results show that a compromise between mentioned objectives and control energy, even in the presence of uncertainties, can be easily made by regulating the control weighting factors.

Keywords: Non-linear suspension system; predictive control; pitch and bounce motions; optimization; MIMO system

1. INTRODUCTION

Vehicle suspension systems play an important role in the performance of modern road vehicles. They are mainly designed to isolate the vehicle and its occupants from the road roughness. This objective is to comply with the ride comfort criterion. On the other hand, a suspension system must ensure good contact between the tires and the road to meet the road holding criterion [1, 2]. The road holding is important because it ensures vehicle safety on rough roads during turning, braking and accelerating. The two mentioned objectives are in conflict with each other and a tradeoff between them is necessary in the design of a suspension system. Another design factor is the restriction of suspension deflection. Large suspension deflection causes excessive pitch and roll motions in abrupt manoeuvers.

A passive suspension consists of constant elasto-damping elements and only provides uniform behaviour. Therefore, it cannot efficiently compromise between the ride comfort and handling in different situations [3]. Designing an active suspension system despite its cost and complexity can highly compensate for the drawbacks of passive systems. One of the main constraints in the design of an active system is power limitation of its actuator, which is required to apply large forces in short durations [4, 5]. This constraint should be considered as an important design factor in the active suspension system [6].

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The first step in the design process of an efficient active suspension system is to select a suitable model referred as design model. The design model should include the main features of a real car suspension system. Various suspension system models in terms of number of degrees of freedom (DOF), complexity and design objective have been used by different researchers. Among them, the quarter-car model is the most common-used design model due to its simplicity. However, it only captures the vertical motion of suspension and does not include the pitch motion which becomes very important in vehicle manoeuvers. Therefore, using a half-car suspension model considering the pitch motion is preferred as an appropriate model in studying the behaviour of road vehicles. It should be noted that, designing the controller based on a comprehensive vehicle model with large degrees of freedom is often mathematically complicated.

Another important factor to achieve a more accurate model is choosing realistic elasto-damping models for suspension elements. Kim and Ro [7] compared the responses of linear and non-linear suspension models for different road inputs and concluded that the non-linear characteristics of suspension elements should be considered in developing a more accurate system model. There are many studies that have used a half-car model but with linear elements [8-11]. Generally, a non-linear half-car model considering pitch and bounce motions needs to be developed for the design of active suspension systems.

Beside the dynamic models highlighted above, different control strategies have been reported for active suspension design. Commonly used control approaches of suspension systems include sliding mode control [7, 12-14], non-linear adaptive control [9, 15], fuzzy logic control [8, 15, 16, 17], back-stepping control [18, 19], multi-objective robust control [20, 21], H_∞ control [4, 22, 23], LQ-based control [24], LQG control approach [25, 26] and feedback linearization control [27]. Each of these control methods considers some aspect of active suspension problem. However, as mentioned before, the main control problem for the active suspension involves finding a suitable external force that trades off between conflicting objectives. Additionally, the control law must minimize the required injected forces. Thus, an optimal control problem should be considered to find the control law by minimizing the weighted combination of conflicting objectives as performance index. On the other hand, the use of a non-linear model for suspension elements can broaden the valid range of operation as well as improve the accuracy of the model.

Classic optimal control strategies based on linear models have been widely used to design active suspension controllers [6, 10, 28-30]. Although these methods are systematic and can provide a good compromise between conflicting objectives and control force, they are exclusively applicable to linear models, whereas a real vehicle suspension has intrinsic non-linear behaviour. There are some non-linear control approaches which have used the optimization in controller design [31-33]. However, they have used numerical or search methods for calculating the control input at each instant of control process. In general, the application of classic optimal control theories for non-linear systems requires solution of the derived non-linear differential equations. There are no analytical solutions for all these equations. In addition, numerical computation methods require online dynamic optimization which is difficult to implement [34, 35].

In this study, a predictive control approach is employed to design a non-linear MIMO controller for active suspension system using an optimization process. The method directly employs non-linear elasto-damping elements and analytically leads to a closed form control law which is mathematically tractable and suitable for implementation. In addition, online numerical computations are not required at each sample time. Following the success of this method fora quarter-car suspension model which is related to a single input system [2], the authors decided to extend this approach for a realistic model considering pitch and bounce motions. In this case, a MIMO control problem with two inputs is solved analytically. To investigate the special features of the derived control laws, the closed-loop system dynamics are evaluated and simulation results are presented.

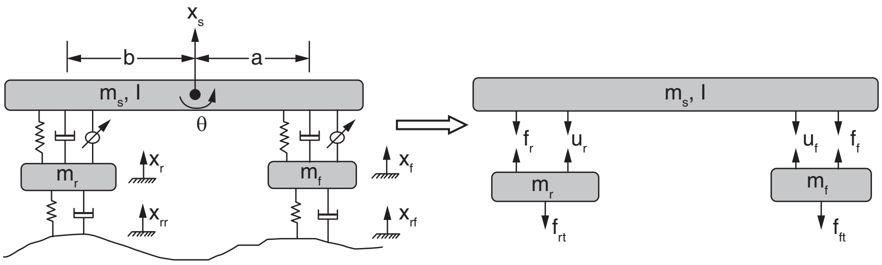

Figure 1. 4DOF half-car model and its free body diagram.

2. SUSPENSION MODEL

In this section, a 4DOF half-car model containing essential features of the vehicle suspension system is proposed for designing the controller. The bounce and pitch motions of the sprung mass together with the front and rear wheels hop constitute the DOF for this model (Figure 1). The equations of motion for the model are simply derived using Newton's second law as follows:

For the sprung mass:

$$m_s\ddot{x}_s = -f_f - f_r + u_f + u_r \tag{1}$$

$$I\ddot{\theta} = -af_f + bf_r + au_f - bu_r \tag{2}$$

and for the un-sprung mass

$$m_f\ddot{x}_f = f_f - f_{ft} - u_f \tag{3}$$

$$m_r\ddot{x}_r = f_r - f_{rt} - u_r \tag{4}$$

where x_s is the sprung mass displacement, x_f and x_r are the front and rear tire displacements, respectively, f_f and f_r represent the front and rear elasto-damping forces, respectively. Also, m_s, m_f and m_r denote the sprung mass, the front un-sprung mass and the rear un-sprung mass, respectively. f_{ft} and f_{rt} are the visco-elastic forces of front and rear tires, respectively. u_f and u_r represent the front and rear actuator forces, respectively. a and b are respectively the front and rear suspension distances from vehicle center of mass, I is the sprung mass moment of inertia about the body centroidal axis and θ is the pitch rotation angle of the sprung mass.

In linear suspension models, the spring and damper forces are often described as linear functions of strut deflection and relative velocity. However, the linear behaviour of suspension spring and damper is valid only for small displacements from their equilibrium position. For example, the spring has a hardening behaviour for deflections beyond the linear region. Kim and Ro [7] proposed polynomial functions for elasto-damping forces. They obtained these functions by fitting polynomial curves as follows to the measured data:

$$f_{sp} = k_1\Delta x + k_2\Delta x^2 + k_3\Delta x^3 \tag{5}$$

$$f_d = c_1\Delta\dot{x} + c_2\Delta\dot{x}^2 \tag{6}$$

where f_{sp} and f_d are spring and damper forces respectively, and Δx and $\Delta\dot{x}$ represent suspension deflection and velocity, respectively. k_1, k_2 and k_3 are stiffness coefficients and c_1 and c_2 are damping coefficients.

The state variables are defined as

$$x_1 = x_s + a\theta - x_f \tag{7}$$

$$x_2 = \dot{x}_s + a\dot{\theta} \tag{8}$$

$$x_3 = x_s - b\theta - x_r \tag{9}$$

$$x_4 = \dot{x}_s - b\dot{\theta} \tag{10}$$

$$x_5 = x_f - x_{rf} \tag{11}$$

$$x_6 = x_r - x_{rr} \tag{12}$$

$$x_7 = \dot{x}_f \tag{13}$$

$$x_8 = \dot{x}_r \tag{14}$$

where x_1 and x_2 are the suspension deflection and absolute velocity of sprung mass in front station, x_3 and x_4 are the suspension deflection and absolute velocity of sprung mass in rear station, x_5 and x_6 are the front and rear tire deflections, and x_7 and x_8 are the front and rear tire velocities, respectively. x_{rf} and x_{rr} are the road elevation in front and rear stations, respectively.

Now, by defining the above state variables, the front and rear suspension and tire forces are obtained as:

$$f_f = k_{f3}x_1^3 + k_{f2}x_1^2 + k_{f1}x_1 + c_{f2}\left(x_2 - x_7\right)^2 + c_{f1}\left(x_2 - x_7\right) \tag{15}$$

$$f_r = k_{r3}x_3^3 + k_{r2}x_3^2 + k_{r1}x_3 + c_{r2}\left(x_4 - x_8\right)^2 + c_{r1}\left(x_4 - x_8\right) \tag{16}$$

$$f_{ft} = k_{uf}x_5 + c_{uf}\left(x_7 - \dot{x}_{rf}\right) \tag{17}$$

$$f_{rt} = k_{ur}x_6 + c_{ur}\left(x_8 - \dot{x}_{rr}\right) \tag{18}$$

where k_{f1}, k_{f2}, k_{f3} and c_{f1}, c_{f2} are stiffness and damping coefficients for front elasto-damping elements. k_{r1}, k_{r2}, k_{r3} and c_{r1}, c_{r2} are stiffness and damping coefficients for rear elasto-damping elements. The coefficients k_{uf} and c_{uf} denote the stiffness and damping of the front tire and the coefficients k_{ur} and c_{ur} denote the stiffness and damping of the rear tire, respectively.

The road input for the rear wheel is a delayed form of the road input for the front wheel:

$$x_{rr} = x_{rf}\left(t + \tau\right) \tag{19}$$

Here, τ is the time delay between the front and rear wheels in crossing over the road and depends on the wheel base and speed of the vehicle:

$$\tau = \frac{a+b}{v} \tag{20}$$

where υ is the vehicle speed. Now, the state space equations are obtained by substituting the state variables from eqns. (7) - (14) into eqns. (1) - (4):

$$\dot{x}_1 = x_2 - x_7 \tag{21}$$

$$\dot{x}_2 = g_2 + d_1 u_f + d_2 u_r \tag{22}$$

$$\dot{x}_3 = x_4 - x_8 \tag{23}$$

$$\dot{x}_4 = g_4 + d_3 u_f + d_4 u_r \tag{24}$$

$$\dot{x}_5 = x_7 - \dot{x}_{rf} \tag{25}$$

$$\dot{x}_6 = x_8 - \dot{x}_{rr} \tag{26}$$

$$\dot{x}_7 = g_7 - \frac{1}{m_f} u_f \tag{27}$$

$$\dot{x}_8 = g_8 - \frac{1}{m_r} u_r \tag{28}$$

where the terms g_i (i = 2,4,7,8) and d_i (i = 1, …, 4) have been defined as:

$$g_2 = \frac{1}{m_s}\left(-f_f - f_r\right) + \frac{a}{I}\left(-f_f a + f_r b\right) \tag{29}$$

$$g_4 = \frac{1}{m_s}\left(-f_f - f_r\right) - \frac{b}{I}\left(-f_f a + f_r b\right) \tag{30}$$

$$g_7 = \frac{1}{m_f}\left(f_f - f_{ft}\right) \tag{31}$$

$$g_8 = \frac{1}{m_r}\left(f_f - f_{rt}\right) \tag{32}$$

$$d_1 = \frac{1}{\mathrm{m_s}} + \frac{\mathrm{a}^2}{\mathrm{I}} \tag{33}$$

$$d_2 = d_3 = \frac{1}{\mathrm{m_s}} - \frac{\mathrm{ab}}{\mathrm{I}} \tag{34}$$

$$d_4 = \frac{1}{\mathrm{m_s}} + \frac{\mathrm{b}^2}{\mathrm{I}} \tag{35}$$

3. CONTROL SYSTEM DESIGN

In active suspension systems, the controller is designed to make a suitable trade-off between the conflicting objectives with the restricted control forces. In this study, the conflicting objectives are the front and rear sprung mass accelerations, $\dot{x}_2$ and $\dot{x}_4$, the front and rear suspension deflections, x_1 and x_3, and the front and rear tire deflections x_5 and x_6. The mentioned trade-off between these objectives could be achieved by employing an adjustable control law which is presented and discussed in the next sections.

3.1. Development of the control law

In this section, an optimal control law is derived based on predicting the suspension system responses from the non-linear state equations of the half-car model. According to the suspension system requirements described previously, the system outputs are assumed as:

$$y = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix} \tag{36}$$

It should be noted that the sprung mass absolute velocities, x_2 and x_4 are considered as controlled variables and a measure of ride comfort instead of the sprung mass acceleration [2, 6, 7, 26, 29, 30]. This is because the system degree relative to the sprung mass acceleration is zero, whereas, the relative degree of each predicted output has to be at least one. It will be shown in simulation studies that the sprung mass acceleration is perfectly controlled by controlling the absolute velocity.

Now, according to the proposed control method, the non-linear response of each output, x_i (i=1,...,6), for the next time interval, $x_1(t + h)$, where h denotes the prediction time, is first predicted by a Taylor series expansion and then the current control forces, $u(t) = [u_f\ u_r]^T$, are found based on continuous minimization of the predicted responses.

To achieve the compromise between conflicting objectives, a performance index which penalizes the regulation error of output responses in the next instance and the current control expenditure is defined as:

$$J = \frac{1}{2}\sum_{i=1}^{6}\rho_i x_i^2\left(t+h\right) + \frac{1}{2}\rho_7 u_f^2 + \frac{1}{2}\rho_8 u_r^2 \tag{37}$$

where $\rho_i > 0$ and ρ_7 and $\rho_8 \geq 0$ are weighting factors indicating the relative importance of the corresponding terms. The predicted response for each output in the next interval, $x_i(t + h)$ (i=1, …,6), is approximated by a k_ith-order Taylor series expansion at t as follows:

$$x_i\left(t+h\right) = x_i\left(t\right) + h\dot{x}_i\left(t\right) + \ldots + \frac{h^k}{k!}x_i^k\left(t\right) \quad \mathrm{i} = 1,\cdots,6 \tag{38}$$

The expansion order, k_i, for each output is selected to be suitable to design the controller based on prediction. To achieve a small control effort and to prevent complexity in derivation and implementation of the control law, the expansion order is limited to the relative degree of the non-linear system [34, 35]. This selection which is related to zero control order, eliminates the derivatives of the control input in the prediction of each output and results in adequate performance for the non-linear system with a lower relative degree [2, 13]. Thus, the control forces will be constant in the prediction interval:

$$\frac{\partial}{\partial\tau}u_f\left(t+\tau\right) = 0 \qquad \mathit{for}\ \tau \in \left[0,h\right] \tag{39}$$

$$\frac{\partial}{\partial\tau}u_r\left(t+\tau\right)=0 \qquad for\ \tau\in\left[0,h\right] \tag{40}$$

The relative degree is determined as the lowest order of the derivative of each output, $x_i(t)$ (i = 1, …,6), in which the input $u_f(t)$ or $u_r(t)$ appears explicitly. According to the state space eqns. (21)-(28), the relative degree of freedom with respect to x_1, x_3, x_5 and x_6 is r = 2 and with respect to x_2 and x_4 is r = 1. Thus, a first-order Taylor series expansions are sufficient for x_2 and x_4 but a second-order Taylor series expansions are required for x_1, x_3, x_5 and x_6. So, the non-linear response of each output for the next time interval is predicted as follows:

$$x_i\left(t+h\right)=x_i\left(t\right)+h\dot{x}_i\left(t\right)+\frac{h^2}{2!}\ddot{x}_i\left(t\right) \qquad \left(i=1,3,5,6\right) \tag{41}$$

$$x_i\left(t+h\right)=x_i\left(t\right)+h\dot{x}_i\left(t\right) \qquad \left(i=2,4\right) \tag{42}$$

Substituting the state space equations and state variables derivatives into the corresponding $x_i(t + h)$ yields:

$$x_i\left(t+h\right)=G_i+l_iu_f+m_iu_r \qquad \left(i=1,\ldots,6\right) \tag{43}$$

where G_is, l_is and m_is (i = 1, …,6) are as follows:

$$G_1=x_1+h\left(x_2-x_7\right)+\frac{h^2}{2!}\left(g_2-g_7\right) \tag{44}$$

$$G_2=x_2+hg_2 \tag{45}$$

$$G_3=x_3+h\left(x_4-x_8\right)+\frac{h^2}{2!}\left(g_4-g_8\right) \tag{46}$$

$$G_4=x_4+hg_4 \tag{47}$$

$$G_5=x_5+h\left(x_7-\dot{x}_{rf}\right)+\frac{h^2}{2!}\left(g_7-\ddot{x}_{rf}\right) \tag{48}$$

$$G_6=x_6+h\left(x_8-\dot{x}_{rr}\right)+\frac{h^2}{2!}\left(g_8-\ddot{x}_{rr}\right) \tag{49}$$

$$l_1=\frac{h^2}{2!}\left(d_1+\frac{1}{m_f}\right) \tag{50}$$

$$m_1=\frac{h^2d_2}{2} \tag{51}$$

$$l_2=hd_1 \tag{52}$$

$$m_2=hd_2 \tag{53}$$

$$l_3 = \frac{h^2 d_3}{2} \tag{54}$$

$$m_3 = \frac{h^2}{2}\left(d_4 + \frac{1}{m_r}\right) \tag{55}$$

$$l_4 = hd_3 \tag{56}$$

$$m_4 = hd_4 \tag{57}$$

$$l_5 = -\frac{h^2}{2m_f} \tag{58}$$

$$m_5 = 0 \tag{59}$$

$$l_6 = 0 \tag{60}$$

$$m_6 = -\frac{h^2}{2m_r} \tag{61}$$

After substituting eqn. (43) into the performance index (37), the following necessary conditions for optimality are applied:

$$\frac{\partial J}{\partial u_f} = 0 \tag{62}$$

$$\frac{\partial J}{\partial u_r} = 0 \tag{63}$$

which lead to:

$$\sum_{i=1}^{6} \rho_i \left(G_i + l_i u_f + m_i u_r\right) l_i + \rho_7 u_f = 0 \tag{64}$$

$$\sum_{i=1}^{6} \rho_i \left(G_i + l_i u_f + m_i u_r\right) b_i + \rho_8 u_r = 0 \tag{65}$$

Solving the above equations for u_f and u_r yields:

$$u_f = \frac{e_2 \sum_{\mathrm{i}=1}^{6} \rho_{\mathrm{i}} \mathrm{G}_{\mathrm{i}} \mathrm{m}_{\mathrm{i}} - e_3 \sum_{\mathrm{i}=1}^{6} \rho_{\mathrm{i}} \mathrm{G}_{\mathrm{i}} \mathrm{l}_{\mathrm{i}}}{e_1 e_3 - \mathrm{e}_2^2} \tag{66}$$

$$u_r = \frac{e_2 \sum_{\mathrm{i}=1}^{6} \rho_{\mathrm{i}} \mathrm{G}_{\mathrm{i}} \mathrm{l}_{\mathrm{i}} - e_1 \sum_{\mathrm{i}=1}^{6} \rho_{\mathrm{i}} \mathrm{G}_{\mathrm{i}} \mathrm{m}_{\mathrm{i}}}{e_1 e_3 - \mathrm{e}_2^2} \tag{67}$$

where e_1, e_2 and e_3 are defined as follows:

$$e_1 = \sum_{i=1}^{6} \rho_i l_i^2 + \rho_7 \tag{68}$$

$$e_2 = \sum_{i=1}^{6} \rho_i l_i m_i \tag{69}$$

$$e_3 = \sum_{i=1}^{6} \rho_i m_i^2 + \rho_8 \tag{70}$$

3.2. Evaluation of the control laws

The main contribution of the present study is derivation of the analytical optimal solution for the MIMO control problem of non-linear active suspension system. In this section, the main features of the derived closed-form control laws (66) and (67) which are mathematically tractable and suitable for implementation are described further.

As mentioned before, the control purpose is to compromise between some conflicting objectives. The importance of each objective is emphasized by assigning a higher amount of its weighting factor relative to the others. Special cases of the control laws (66) and (67) can be obtained when only a specific objective is to be taken into consideration. Therefore, by setting other weighting factors to zero except ρ_i and then by substituting eqns. (68)- (70) into the general case of the control laws (66) and (67), the following especial versions of the control laws are derived.

$$u_f = -\frac{\rho_8 \rho_i l_i}{\rho_7\left(\rho_i m_i^2 + \rho_8\right) + \rho_8 \rho_i l_i^2} G_i \qquad \left(i = 1,\ldots,6\right) \tag{71}$$

$$u_r = -\frac{\rho_7 \rho_i m_i}{\rho_8\left(\rho_i l_i^2 + \rho_7\right) + \rho_7 \rho_i m_i^2} G_i \qquad \left(i = 1,\ldots,6\right) \tag{72}$$

To investigate more specific forms of eqns. (71)-(72), the ride comfort criterion i.e. minimizing the responses x_i(i = 2,4), as a case study are considered in details. However, other responses could be analysed in a similar way. Substituting corresponding G_is from eqns. (44)–(49) into eqns. (71) and (72) yields:

$$u_f = -\frac{\kappa_i}{h}\left(x_i + hg_i\right) \qquad \left(i = 2,4\right) \tag{73}$$

$$u_r = -\frac{\lambda_i}{h}\left(x_i + hg_i\right) \qquad \left(i = 2,4\right) \tag{74}$$

where κ_i and λ_i are definedby substituting corresponding l_is and m_is from eqns. (50)-(61) into eqs (71) and (72):

$$\kappa_i = \frac{\rho_8 \rho_i h^2 d_{i-1}}{\rho_7\left(\rho_i h^2 d_i^2 + \rho_8\right) + \rho_8 \rho_i h^2 d_{i-1}^2} \ (\mathrm{i}=2,4) \tag{75}$$

$$\lambda_i = \frac{\rho_7 \rho_i h^2 d_i}{\rho_8\left(\rho_i h^2 d_{i-1}^2 + \rho_7\right) + \rho_7 \rho_i h^2 d_i^2} (\mathrm{i} = 2,4) \tag{76}$$

The special version of the control laws (73) and (74) are based on only the ride comfort criterion without attention to the other criteria and constraints.

According to eqns. (22)-(24), the system dynamics for $x_i(t)$ i = (2,4) are as follows:

$$\dot{x}_i = g_i + d_j u_f + d_{j+1} u_r \ \left(i = 2,4\right) \tag{77}$$

Substituting the control inputs (73) and (74) into the open-loop eqn (77) leads to the closed-loop dynamics as follows:

$$\dot{x}_i + \frac{1}{h}\left(d_j \kappa_i + d_{j+1}\lambda_i\right)x_i = g_i\left[1 - h\left(d_j \kappa_i - d_{j+1}\lambda_i\right)\right] \ \left(i = 1,2\right) \tag{78}$$

To investigate more specific behaviour of closed-loop dynamics (78), the cheap control case i.e. $\rho_7 = \rho_8 = 0$ is considered. In this case, according to eqns. (75) and (76):

$$d_j \kappa_i + d_{j+1}\lambda_i = 1 \tag{79}$$

$$d_j \kappa_i - d_{j+1}\lambda_i = 0 \tag{80}$$

So, eqn. (78) for the cheap control case takes the following form:

$$\dot{x}_i + \frac{1}{h}x_i = 0 \tag{81}$$

From the above equation, $x_i(t) = 0$ for any $h > 0$. This shows that when there is no limitation in using of the applied control forces, the perfect response of zero for the sprung mass motion is achieved by ignoring the other criteria and constraints.

4. SIMULATION RESULTS

In this section, simulation results of the non-linear half-car suspension system described in the previous sections are presented to show the effectiveness of the proposed control system. Also, the influence of control weighting factors on the system responses and control energy are investigated. The physical values of vehicle suspension parameters are presented in Table 1. These values are related to a typical light commercial car and have been selected near the values of different half car models reported in prevalent papers [7, 9, 14].

To simulate the system responses, the road roughness input shown in Figure 2 is assumed as several continuous bumps with the following equation:

$$x_r = 0.02\left(1 - sin\left(5\pi t\right)\right) - 0.04\sin\left(10\pi t\right), \ 0.1 \ s < t < 1.3 \ s \tag{82}$$

The road input for the rear suspension is assumed to have τ second delay according to equation (20).

To show the adjustability of the controller and the conflictive behaviour of the responses, different sets of weighting factors as free parameters are chosen. Firstly, the ride comfort criterion, as the primary objective, is investigated by selecting the weighting factors as ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = \rho_1 = \rho_3 = \rho_5 = \rho_6 = 0$). This case which is illustrated in Figure 3, which clearly shows how the accelerations of the front and rear suspension stations are perfectly regulated where the sprung mass motion is the

Table I.
Parameters for the case study suspension system

Sprung mass	m_s	500 kg
Front unsprung mass	m_f	45 kg
Rear unsprung mass	m_r	40 kg
Body moment of inertia	I	1200 kgm^2
Distance from the front tire to centroid point	a	1.25m
Distance from the rear tire to centroid point	b	1.45m
Tire stiffness	k_{us}	190000 N/m
Tire damping	c_{us}	70 Ns/m
1st Front spring coefficient	k_{f1}	12394 N/m
2nd Front spring coefficient	k_{f2}	–73696 N/m^2
3rd Front spring coefficient	k_{f3}	3170400 N/m^3
1st Front damper coefficient	c_{f1}	1000 Ns/m
2nd Front damper coefficient	c_{f2}	500 Ns^2/m^2
1st rear spring coefficient	k_{r1}	12150 N/m
2nd rear spring coefficient	k_{r2}	–72500 N/m^2
3rd rear spring coefficient	k_{r3}	3171200 N/m^3
1st Front damper coefficient	c_{r1}	950 Ns/m
2nd Front damper coefficient	c_{r2}	470 Ns^2/m^2
Prediction time	h	0.1s

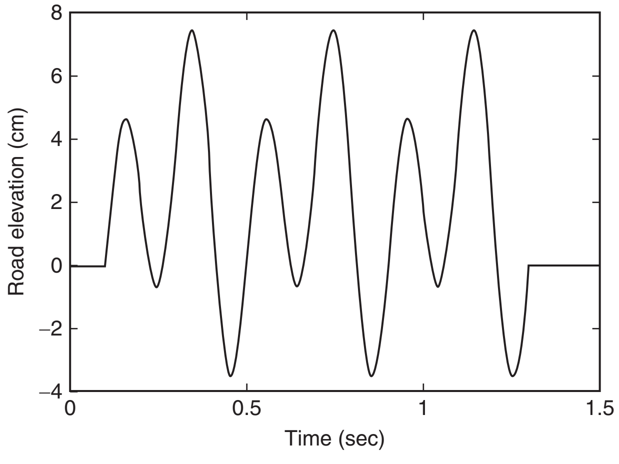

Figure 2. Road profile.

only weighted parameter in the cost function. As a result, the pitch angle magnitude is extremely decreased. However, the responses of tire deflections and suspension deflections do not converge to steady states in finite time, and they remain severely oscillatory. In other words, the road stability of the vehicle is unsafe in this strategy. Also, the control forces become large and oscillatory. It is concluded that the achievement of best ride comfort is provided at the cost of losing the road holding and excessive control forces. Note that, the responses of front and rear suspension stations for the uncontrolled system (dotted line shown in Figure 3), are slightly different in amplitude and settling time because of the delay and the load transfer due to hitting the bumps.

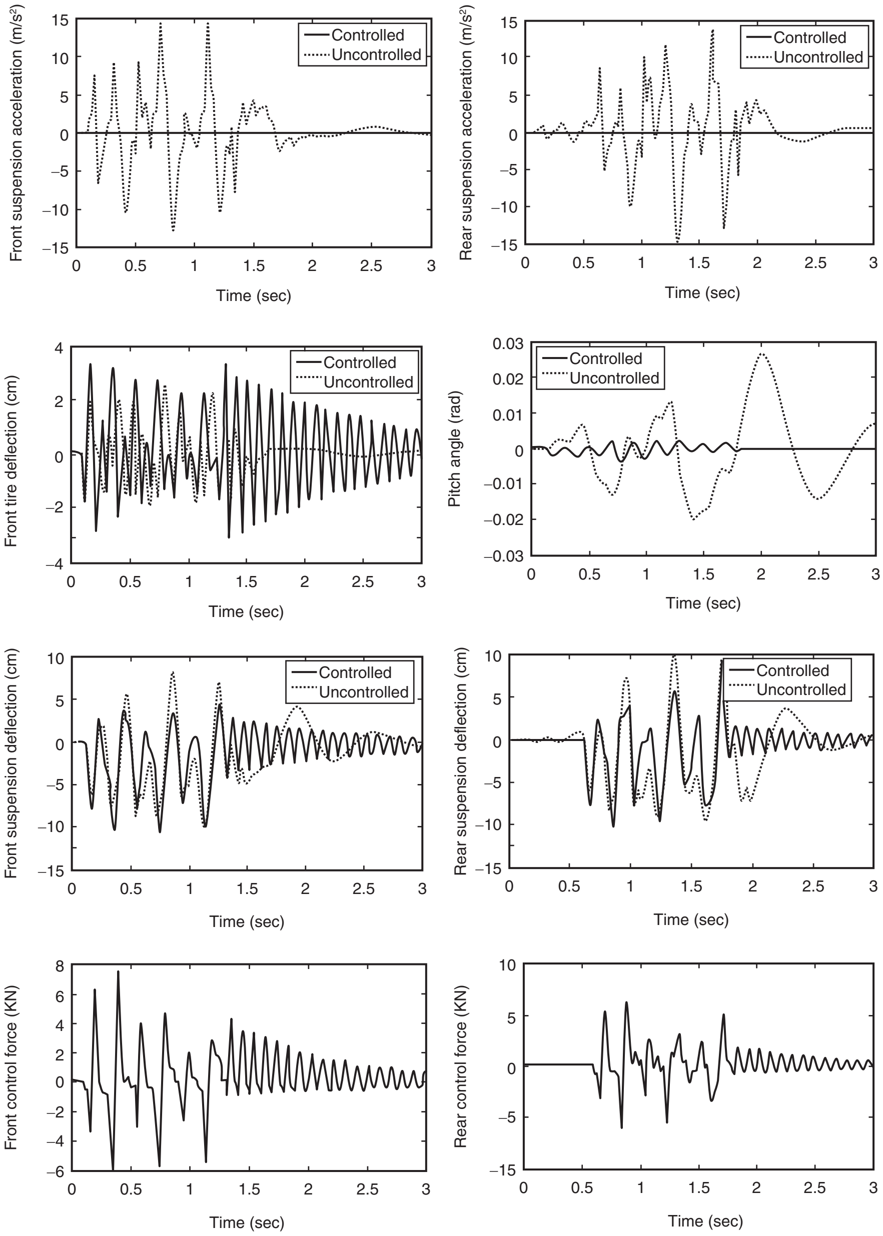

Figure 3. Responses of the suspension system when the ride comfort is the only design criterion ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 0$; $\rho_1 = \rho_3 = \rho_5 = \rho_6 = 0$).

As the second case of simulation study, the control inputs are forced to be limited in the allowable range by weighting the front and rear control forces in the cost function. In this case, a new set of weighting factors ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 10^{-8}$; $\rho_1 = \rho_3 = \rho_5 = \rho_6 = 0$) is considered to limit the control force under the maximum value (e.g. 3.5KN). In this case, according to Figure 4, the accelerations cannot be regulated exactly with limited control forces and the control system cannot achieve the ideal ride comfort. However, the reduction of acceleration and also the pitch angle is still acceptable for the ride comfort in comparison to the uncontrolled or passive suspension case. The selected weighting factors perform the best possible ride comfort in the presence of control force constrains. But, the tire deflection responses do not converge to steady states in finite time, and remain severely oscillatory as in the previous case. In other words, the road stability of the vehicle is still unsafe. This causes

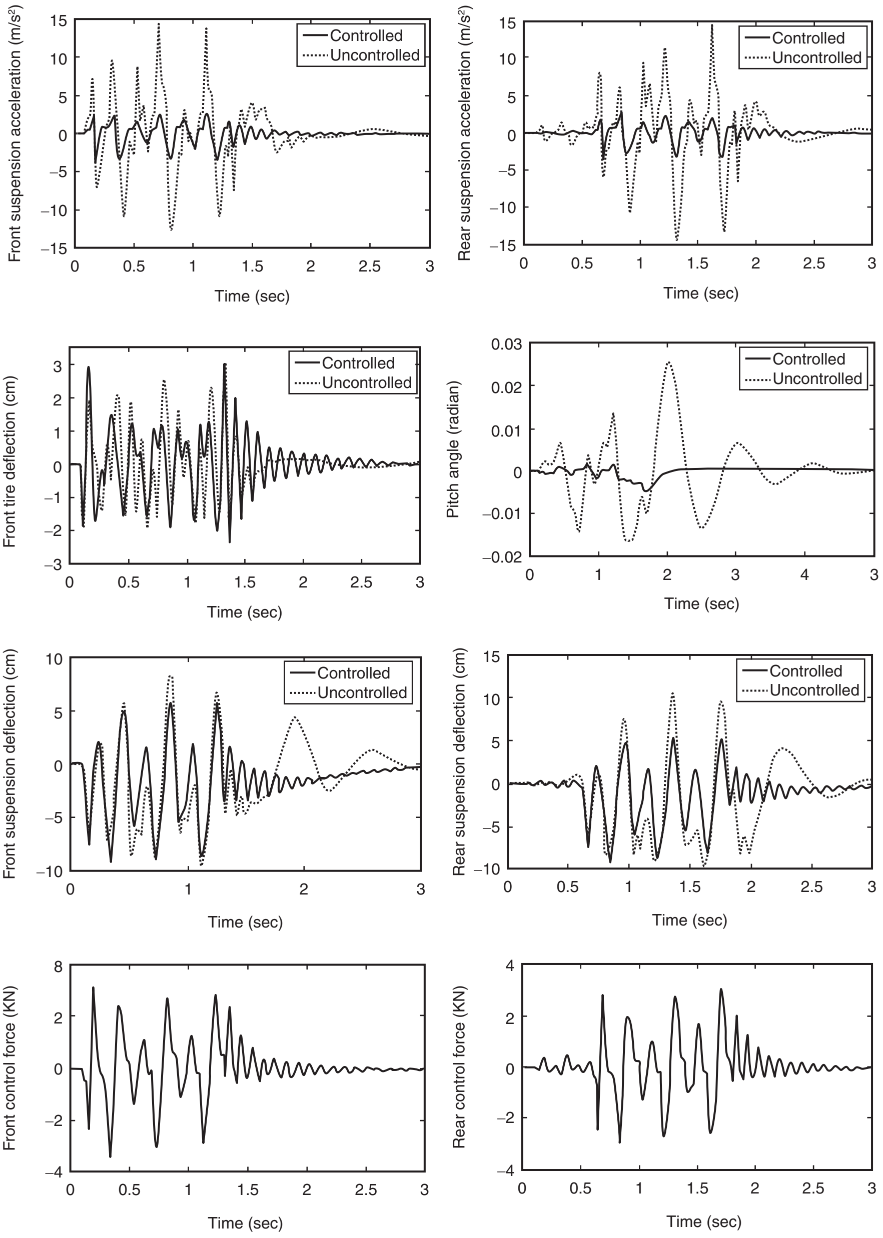

Figure 4. Responses of the suspension system when the ride comfort with limited control forces is the design criterion ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 10^{-8}$; $\rho_1 = \rho_3 = \rho_5 = \rho_6 = 0$).

the control inputs to be oscillatory as well. In order to remove the oscillations and improve the road holding criterion, the tire deflection and suspension deflection of both stations should be weighted in the control law. In this way, a new set of weighting factors ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 5 \times 10^{-8}$; $\rho_1 = \rho_3 = 5 \quad 10^{-7}$; $\rho_5 = \rho_6 = 50$) was selected. The obtained results in Figure 5 indicate that the suspension acceleration and the tire deflection are simultaneously controlled in comparison with passive system. Also, the control forces are under the specified maximum value (3.5KN). As it is seen, the maximum amplitude of the suspension acceleration was reduced to nearly half of the uncontrolled responses in spite of applied practical constraints. Therefore, a suitable trade-off between the control objectives in terms of ride comfort as primary objective and road holding as secondary objective is achieved by considering the restriction on front and rear control forces.

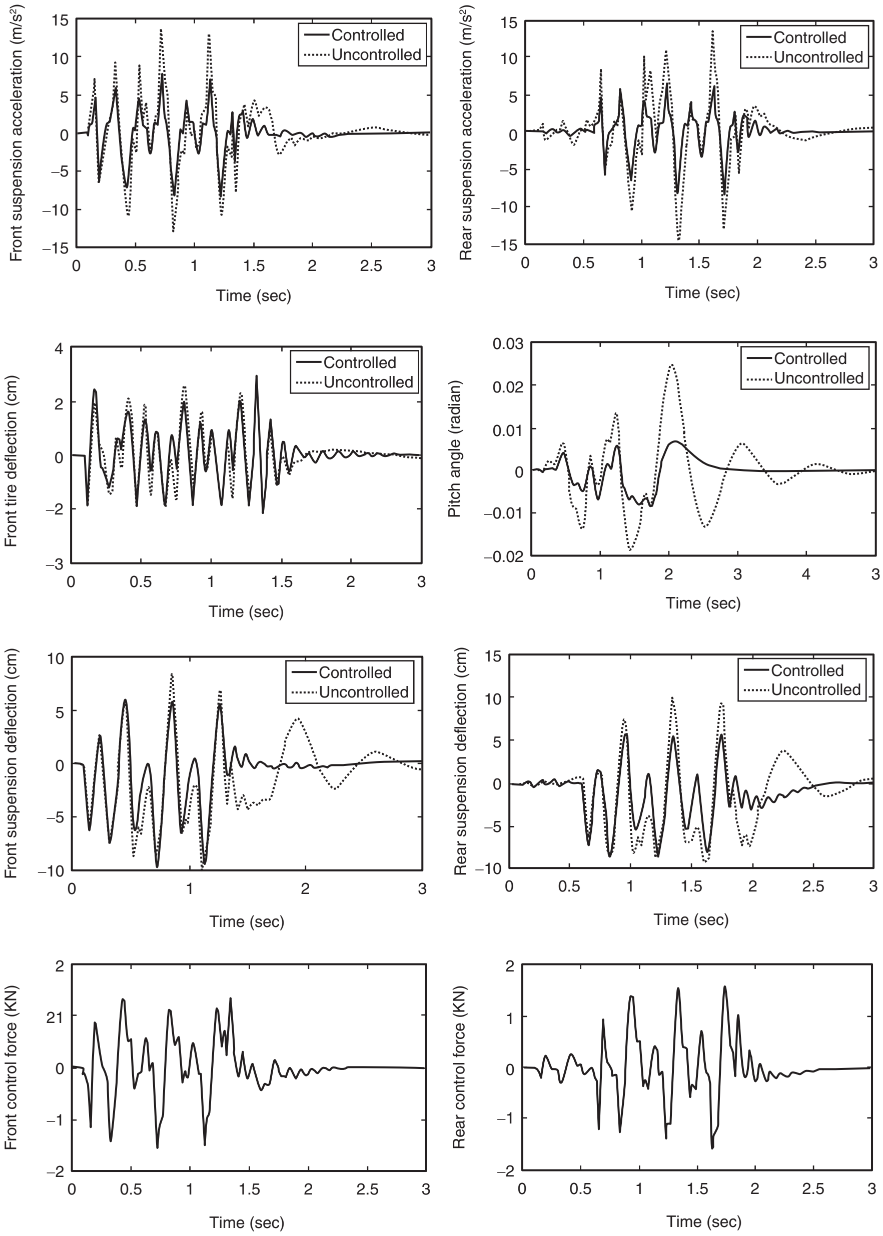

Figure 5. Responses of the suspension system considering all design criteria ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 5 \times 10^{-8}$; $\rho_1 = \rho_3 = 5 \times 10^{-7}$; $\rho_5 = \rho_6 = 50$).

To show robust performance of the designed controller, the last set of weighting factors has been applied to the controller considering some uncertainties in the system parameters. The results of Figure 7 clearly show that the performance of the controller did not decrease considerably through uncertainty. In other words, the controller has a suitable robustness in the presence of uncertainty.

The root mean square (RMS) values for all responses from Figures 3 to 6 are also presented in Table 2 to offer a quantitative insight into changes in responses.

Finally, in order to show the results in the frequency domain, the impulse response of suspension system is shown in Figure 7. In this case, the road profile is modeled as a Dirac delta function and the responses in frequency domain are obtained by taking fast Fourier transform (FFT) of each output in the time domain. Since the system under investigation is non-linear, applying purely frequency domain analysis will not reveal all the system characteristics unlike linear systems.

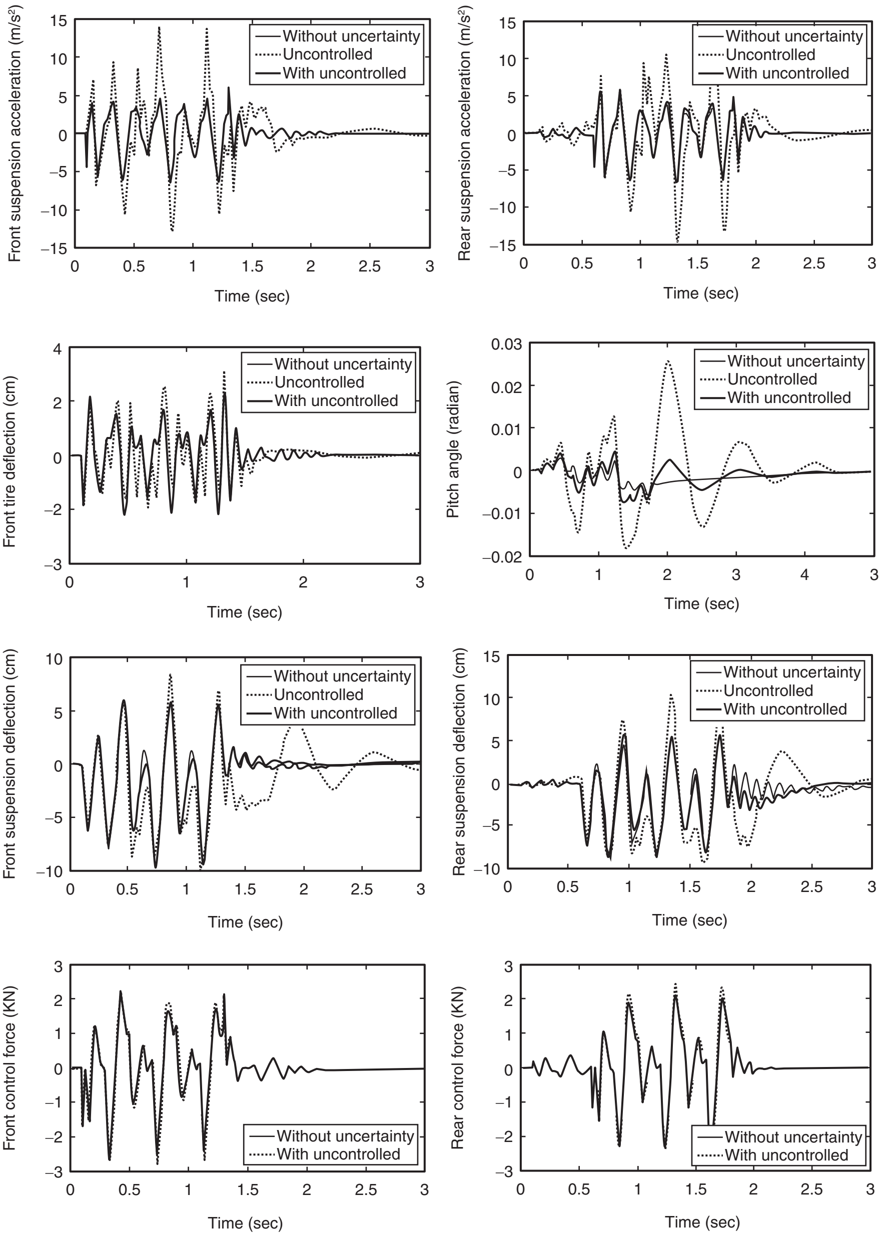

Figure 6. Responses of the suspension system in the presence of uncertainties ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 5 \times 10^{-8}$; $\rho_1 = \rho_3 = 5 \times 10^{-7}$; $\rho_5 = \rho_6 = 50$).

However, the results of Figure 7 indicate the improved performance of the controlled system in a wide range of frequencies.

5. CONCLUSION

In this paper, a predictive control approach has been developed to control a half-car suspension system model with non-linear elasto-damping forces. Using an optimization process, a MIMO non-linear control law has been developed in a closed-form which is mathematically tractable and suitable for implementation. The results show that a compromise between conflicting objectives in the presence of control force limitations can be easily made by regulation of control weighting factors.

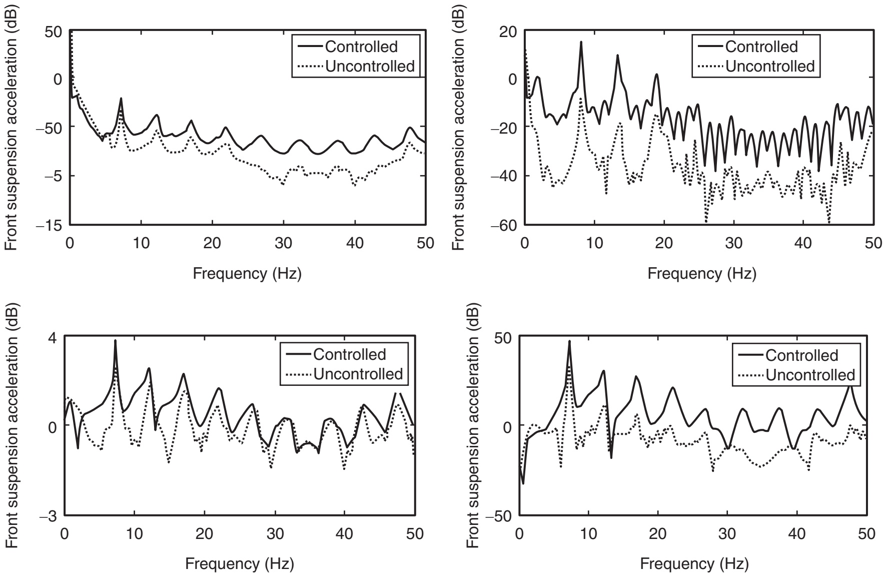

Figure 7. Impulse responses of suspension system in frequency domain ($\rho_2 = \rho_4 = 1$; $\rho_7 = \rho_8 = 5 \times 10^{-8}$; $\rho_1 = \rho_3 = 5 \times 10^{-7}$; $\rho_5 = \rho_6 = 50$).

Table 2.

RMS values of the suspension responses $\left(\sqrt{1/T \int_0^T x_i^2 \, dt}\right)$

	Front suspension acceleration ($x_i = \dot{x}_2$)	Rear suspension acceleration ($x_i = \dot{x}_4$)	Front tire deflection ($x_i = \dot{x}_5$)	Pitch angle ($x_i =$)	Front control force ($x_i = u_f$)	Rear control force ($x_i = u_r$)
Uncontrolled	3.7906	4.0823	0.0083	0.0109	——	——
Controlled in Figure 3	1.0289	0.9420	0.0084	0.0013	969.9752	965.9644
Controlled in Figure 4	2.3090	2.2036	0.0076	0.0041	473.4736	503.8095
Controlled in Figure 5	2.4722	2.3682	0.0054	0.0038	470.7252	442.9735
Controlled in Figure 6 with uncertainties	2.4817	2.3805	0.0059	0.0041	475.2138	481.3251

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